Differential Geometry III: Riemannian geometry

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3.1 Let $\mathcal{M}^n$ be a smooth manifold and let $(x^1, \dots, x^n)$ a local system of coordinates around $p \in \mathcal{M}$. Let also $S \in \otimes^k T_p\mathcal{M} \otimes^l T_p^*\mathcal{M}$ be a tensor of type (k,l) at p and let $S^{i_1 i_2 \dots i_k}{}_{j_1 j_2 \dots j_l}$ be its corresponding components. We will define the *contraction* $\mathrm{tr}(S)$ to be the tensor

$$\mathrm{tr}(S) = S^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_l},$$

i.e. the components of $\mathrm{tr}(S)$ in the $(x^1, \dots, x^n)$ coordinates are simply the components of S after summing over the first covariant and contravariant indices. Show that $\mathrm{tr}(S)$ is well-defined *independently* of the choice of coordinate system, i.e. show that if $(y^1, \dots, y^n)$ is a different coordinate system around p and $\tilde{S}^{i_1 i_2 \dots i_k}{}_{j_1 j_2 \dots j_l}$ are the components of S with respect to these coordinates, then

$$\begin{aligned} &S^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_l} \\ &\qquad = \tilde{S}^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial y^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_k}} \otimes dy^{j_2} \otimes \dots \otimes dy^{j_l}. \end{aligned}$$

Remark. In the case when S is of type $(1,1)$, and hence can be viewed as a linear map $S : T_p\mathcal{M} \to T_p\mathcal{M}$, $\mathrm{tr}(S)$ is simply the trace of the matrix representation of S; in that case, the statement of the above exercise reduces to the well-known fact that the trace of a linear automorphism is independent of the choice of basis of vectors.

3.2 Let $\mathcal{M}$ be a smooth manifold of dimension n. In this exercise, we will prove that the tangent bundle $T\mathcal{M} = \cup_{p\in\mathcal{M}} T_p\mathcal{M}$ naturally admits the structure of a manifold of dimension $2n$.

Let $\{\mathcal{U}_\alpha, \phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n\}_\alpha$ be a smooth atlas on $\mathcal{M}$. For any pair $(\mathcal{U}_\alpha, \phi_\alpha)$ in this atlas, let $(x^1, \dots, x^n)$ be the associated system of coordinates; we can define a map

$$\tilde{\phi}_\alpha : T\mathcal{U}_\alpha = \cup_{p\in\mathcal{U}_\alpha} T_p\mathcal{M} \ \to \ \phi_\alpha(\mathcal{U}_\alpha) \times \mathbb{R}^n$$

as follows:

$$\tilde{\phi}_\alpha(p, v) = (\phi(p);\, dx^1(v), \dots, dx^n(v)).$$

We will equip $T\mathcal{M}$ with the topology that makes all these maps homeomorphisms, i.e.:

$$\mathcal{V} \subset T\mathcal{M} \ \text{ is open iff } \ \tilde{\phi}_\alpha\big(\mathcal{V} \cap T\mathcal{U}_\alpha\big) \subset \mathbb{R}^n \times \mathbb{R}^n \text{ is open for all } \alpha.$$

(a) Show that $T\mathcal{M}$ equipped with the above topology is *Hausdorff* and *second countable.*

(b) Show that $\{(T\mathcal{U}_\alpha, \tilde{\phi}_\alpha)\}_\alpha$ constitutes a *smooth* atlas on $T\mathcal{M}$.

(c) Show that the base projection map $\pi : T\mathcal{M} \to \mathcal{M}$ (which acts by $\pi : T_p\mathcal{M} \to p$) is smooth. Moreover, for any smooth vector field $X \in \Gamma(T\mathcal{M})$, show that the map $X : \mathcal{M} \to T\mathcal{M}$ (sending any $p \in \mathcal{M}$ to $X_p \in T_p\mathcal{M}$) is a smooth *immersion.*

3.3 Let X, Y be smooth vector fields on a smooth manifold $\mathcal{M}$. We define the commutator (or *Lie bracket*) $[X, Y]$ of X and Y to be the linear function $[X, Y] : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for all } f \in C^\infty(\mathcal{M}).$$

(a) Show that $[X, Y]$ is a smooth vector field on $\mathcal{M}$.

(b) Show that $[\cdot, \cdot]$ satisfies the following algebraic identities for any $X, Y, Z \in \Gamma(\mathcal{M})$:

1. $[X, Y] = -[Y, X]$ (*anticommutativity*).
2. $[X, aY + bZ] = a[X, Y] + b[X, Z]$ for any constans a, b ($\mathbb{R}$-*linearity*).
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*).

(c) Is $[\cdot, \cdot] : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ a $(1, 2)$-tensor field?

3.4 Let (M, g) be a smooth Riemannian manifold.

(a) For any 1-form ω on $\mathcal{M}$, let us consider the vector field $\omega^\sharp$ defined so that, for any $X \in \Gamma(\mathcal{M})$:

$$g(X, \omega^\sharp) \doteq \omega(X).$$

Compute the components of $\omega^\sharp$ in any local coordinate chart $(x^1, \dots, x^n)$.

(b) We will define the *gradient* of a function $f : \mathcal{M} \to \mathbb{R}$ to be the vector field

$$\nabla f \doteq df^\sharp.$$

Compute the gradient of a function $f : \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates.

(c) We can naturally construct a positive definite and symmetric $(2, 0)$-tensor $\tilde{g}$ acting as an inner product on the space of 1-forms by the formula

$$\tilde{g}(\omega_1, \omega_2) \doteq g(\omega_1^\sharp, \omega_2^\sharp) \quad \text{for all } \omega_1, \omega_2 \in \Gamma^*(\mathcal{M}).$$

Compute the coefficients $\tilde{g}^{ij}$ of $\tilde{g}$ in any local coordinate system as a function of the coefficients of g.

3.5 Let $\mathcal{M}^n$ be a smooth manifold and $\omega : \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$ be a $C^\infty(\mathcal{M})$-linear functional. We will show that ω is in fact an 1-form on $\mathcal{M}$, i.e. if $Y \in \Gamma(\mathcal{M})$ then, for all $p \in \mathcal{M}$, $\big(\omega(Y)\big)(p)$ depends only on $Y|_p$.

(a) Explain why it suffices to show that if Y vanishes at p, then $\big(\omega(Y)\big)(p) = 0$.

(b) Let $\mathcal{U}$ be an open neighborhood of p covered by a coordinate chart $(x^1, \dots, x^n)$. Show that there exists an open neighborhood $\mathcal{V}$ of p contained inside $\mathcal{U}$ and smooth vector fields $\{X_i\}_{i=1}^n$ on $\mathcal{M}$ such that $X_i = \frac{\partial}{\partial x^i}$ on $\mathcal{V}$. (*Hint: Use a suitable cut-off function* $\psi : \mathcal{M} \to [0, +\infty)$ *which is equal to* 1 *in small a neighborhood of* p.)

(c) Show that if $Y|_p = 0$, then there exists a finite number of vector fields $\{V_k\}_k$ such that

$$Y = \sum_k f_k V_k,$$

where the functions $f_k \in C^\infty(\mathcal{M})$ satisfy $f_k(p) = 0$. Deduce that $\omega(Y)(p) = 0$.

The same argument should also work for more general $C^\infty(\mathcal{M})$-multilinear maps $T : \Gamma^*(\mathcal{M}) \times \cdots \times \Gamma^*(\mathcal{M}) \times \Gamma(\mathcal{M}) \times \cdots \times \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$.